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The hyperspace of hereditarily decomposable subcontinua of a cube is the Hurewicz set

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To the memory of Professor J.J. Charatonik

Abstract

The hyperspaces of hereditarily decomposable continua and of decomposable subcontinua without pseudoarcs in the cube of dimension greater than 2 are homeomorphic to the Hurewicz set of all nonempty countable closed subsets of the unit interval [0, 1]. Moreover, in such a cube, all indecomposable subcontinua form a homotopy dense subset of the hyperspace of (nonempty) subcontinua.

MSC: primary 54B20, 57N20; secondary 54F45, 54H05

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1. Introduction

By a *continuum* we mean a compact connected metric space. A continuum is *decomposable* provided that it is the union of two proper subcontinua. The continuum is said to be *hereditarily decomposable* if each of its nondegenerated subcontinua is decomposable. Given a continuum X, we denote by $\mathcal{D}(X)$ and $\mathcal{HD}(X)$ the spaces of all decomposable and all hereditarily decomposable subcontinua of X, respectively.

The Hurewicz set, denoted by $\mathcal{H}$, is the space of all nonempty compact countable subsets of the interval I. It is known that $\mathcal{H}$ is coanalytic complete because it is a Π^1_1-absorber [2, Theorems 1.1 and 1.4].

It is known that $\mathcal{D}(X)$ is an F_σ-subset of $C(X)$ and that the hyperspace of pseudoarcs in a cube I^n is a residual G_δ-set in $C(I^n)$. U. Darji [4] showed that the space $\mathcal{HD}(I^n)$, $n \geqslant 2$, is coanalytic complete. P. Krupski [9] proved that, for $n \geqslant 2$, the hyperspace of subcontinua which do not contain pseudoarcs is a coanalytic complete subset of $C(I^n)$. In the paper we characterize the hyperspaces of hereditarily decomposable subcontinua and of decomposable subcontinua which do not contain pseudoarcs; namely, we demonstrate that, for $n \geqslant 3$, both of them are homeomorphic to $\mathcal{H}$. To achieve this we use a method of absorbers. A part of the proof is showing that, for $n \geqslant 3$, all decomposable continua form a σZ-set in $C(I^n)$, which implies the existence of a deformation of $C(I^n)$ through $C(I^n) \setminus \mathcal{D}(I^n)$.

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2. Preliminaries

All spaces in the paper are metric separable. The hyperspaces 2^X of all nonempty compact subsets of X and $C(X) \subset 2^X$ of all nonempty continua in X are equipped with the Hausdorff metric

$$d_H(K, L) = \inf\{\varepsilon > 0\colon\ K \subset B(L; \varepsilon) \text{ and } L \subset B(K; \varepsilon)\},$$

where $B(A; \varepsilon)$ stands for the open ε-ball about the subset A in X. Recall that 2^{I^k} and, for $k > 1$, $C(I^k)$ are homeomorphic to the Hilbert cube I^∞, where $I = [0, 1]$ and $k \in \mathbb{N} \cup \{\infty\}$ [3,6].

A *chain* is a collection of open sets $\mathcal{C} = \{C_0, C_1, \ldots, C_k\}$ such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. The elements of a chain are called *links*. A chain is called an ε-chain if all its links are of diameter less than ε. A continuum X is said to be *chainable* if, for every $\varepsilon > 0$, there is an ε-chain covering X.

A *pseudoarc* is a nondegenerate, hereditarily indecomposable, chainable continuum. It is known that two pseudoarcs are always homeomorphic [10].

B. Knaster [8] described the pseudoarc as the intersection of a nested family of bands. A *band* is a finite union of rectangles $R_1, \ldots, R_k$ in I^2 such that

- all the sides of R_i's are parallel to the sides of I^2;
- if $|i - j| = 1$ then the intersection of R_i and R_j is their common side;
- if $|i - j| = 2$ then R_i and R_j have at most one common point;
- if $j = i + 2$ and $R_i \cap R_j \neq \emptyset$ then R_{i+1} is called a *corner rectangle*;
- if $|i - j| > 2$ then $R_i \cap R_j = \emptyset$;
- $(\bigcup_{i=1}^k R_i) \cap (\{0\} \times I)$ and $(\bigcup_{i=1}^k R_i) \cap (\{1\} \times I)$ are sides of R_1 and R_k, respectively.

By the *left* (*right*) *side* of a band B we mean the segment $B \cap (\{0\} \times I)$ (or $B \cap (\{1\} \times I)$, respectively). If ∂B is the combinatorial boundary of band B then the complement in ∂B of the union of left and right sides consists of two components. The closures of the components are called *borders* of B; the one which intersects $\{0\} \times I$ in a higher point is called the *upper border*, the other one is the *lower border*.

We say that two polygonal lines are parallel provided that they are borders of a band in I^2.

A band B is *uniform of width* t if its left side is of length t and every corner rectangle is a square. If $t = 0$ then a band is a polygonal line.

In the Knaster's construction of the pseudoarc uniform bands built of squares are considered. In the first step I^2 is divided into 25 congruent squares and the band B_1 consists of some of them. Then, the smaller squares are divided again into 25 congruent squares and the band $B_2 \subset B_1$ is taken as it is shown in Fig. 1.

If $n \geqslant 2$ then every square the band B_n is built of is divided into 25 congruent squares and a band $B_{n+1} \subset B_n$ is composed of some of them according to an algorithm described in [8]. The way of choosing B_{n+1} in B_n is called a *pattern* for B_{n+1}. The intersection $\bigcap_{n=1}^\infty B_n$ is the Knaster pseudoarc.

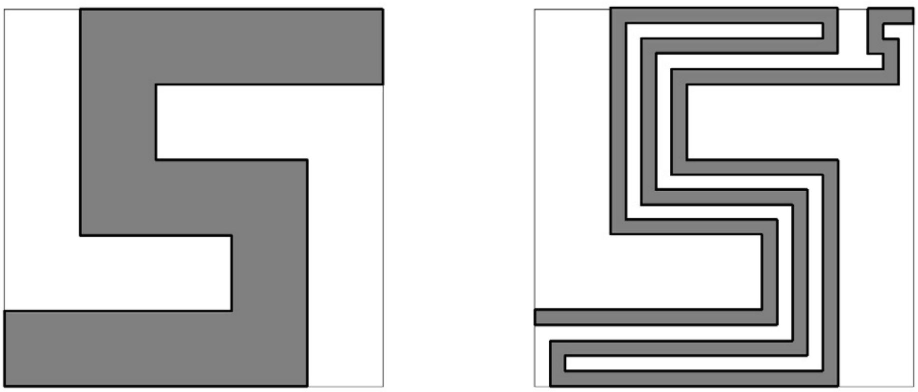

Fig. 1.

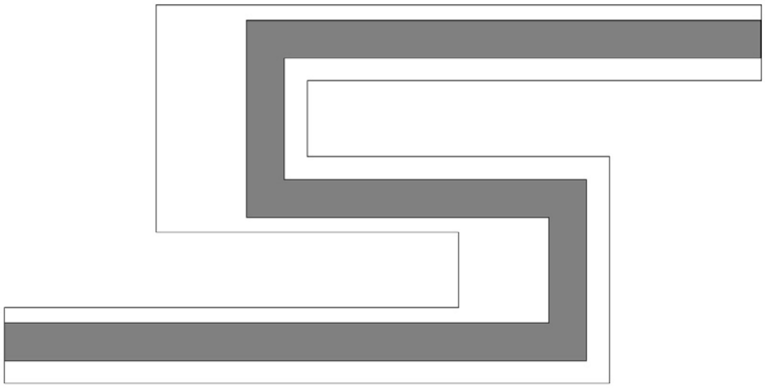

Fig. 2.

The square I^2 in the above construction can be substituted by a rectangle divided into 25 rectangles similar to the first one. In this case the first band K_1 is composed of the small rectangles chosen according to the pattern for B_1. One can modify the next steps of the construction as well. Take an arbitrary uniform band $L_1 \subset K_1$ such that L_1 has the positive width and its borders are parallel to the borders of K_1 (Fig. 2).

Divide L_1 into rectangles so that:

- every corner rectangle is a square;
- the number of rectangles preceding the first corner squares is equal to the number of the squares preceding the first corner square in the Knaster's construction; all of these rectangles are congruent;
- the number of rectangles following the last corner squares is equal to the number of the squares following the last corner square in the Knaster's construction; all of these rectangles are congruent;
- the number of rectangles between every pair of corner squares is equal to the number of the squares between the appropriate corner squares in the Knaster's construction;
- if R_s and R_t are corner rectangles and there is no corner rectangle between them then all rectangles between R_s and R_t are congruent.

Then, divide all rectangles L_1 is built of into 25 rectangles, choose the band K_2 according to the pattern for B_2, inscribe a uniform band $L_2 \subset K_2$ with borders parallel to the borders of K_2 (Fig. 3), and so on. (Observe that, for each n, the band K_{n+1} is strictly determined by the choice of L_n, although L_n can be an arbitrary uniform band inscribed in K_n.) The intersection of the family $\{L_n\}_{n\in\mathbb{N}}$ is a pseudoarc as well.

A subset D of a complete space Z is *analytic* if D is a continuous image of a Borel subset of a complete space; D is *coanalytic* if $Z \setminus D$ is analytic. The set D is (*co*)*analytic complete* if D is (co)analytic and for any (co)analytic

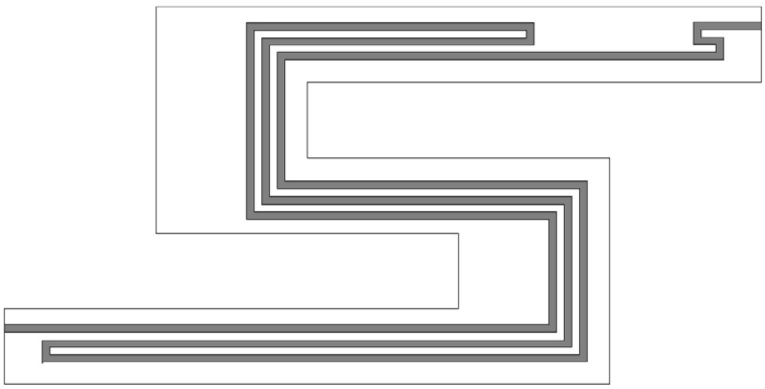

Fig. 3.

subset C of a 0-dimensional complete space Y there is a continuous mapping $\xi : Y \to Z$ (called a *reduction of C to D*) such that $\xi^{-1}(D) = C$ [7]. Clearly, a (co)analytic complete set cannot be Borel.

Given a continuum X, denote by $\mathcal{NP}(X)$ the hyperspace of subcontinua which do not contain pseudoarcs in X.

Lemma 2.1. [4,9] *Let $n \in \mathbb{N} \cup \{\infty\}$, $n \geqslant 2$. Then $\mathcal{HD}(I^n)$ and $\mathcal{NP}(I^n)$ are coanalytic complete.*

Let $\mathbb{N}^{\mathbb{N}}$ be the Baire space of all infinite sequences of natural numbers. The countable set $\mathbb{N}^{<\mathbb{N}}$ of nonempty all finite sequences of natural numbers can be considered as a subspace of $\mathbb{N}^{\mathbb{N}}$. If $\sigma \in \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$, then $|\sigma|$ denotes the length of σ (if $\sigma \in \mathbb{N}^{\mathbb{N}}$ then $|\sigma| = \infty$). If σ is a sequence and $n \leqslant |\sigma|$, then $\sigma \restriction n$ denotes the partial finite sequence of the first n elements of σ. We write $\sigma < \sigma'$ for $\sigma, \sigma' \in \mathbb{N}^{<\mathbb{N}}$ if $|\sigma| < |\sigma'|$ and $\sigma' \restriction |\sigma| = \sigma$. It is known that any analytic set A in a complete space X can be obtained by a Souslin operation, i.e., there exists a function $S : \mathbb{N}^{<\mathbb{N}} \to 2^X$ such that

$$A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} S(\sigma \restriction n) \tag{S}$$

[7, Theorem 25.7]. Moreover, one can make function S to satisfy additional conditions.

Lemma 2.2. [2, Lemma 1.5] *For each analytic subset A of a complete space X, there exists a function $S : \mathbb{N}^{<\mathbb{N}} \to 2^X$ satisfying* (S) *such that $S(\tau) \subset \operatorname{int} S(\tau')$ for $\tau' < \tau$.*

Recall that a closed subset B of a Hilbert cube X is a *Z-set* in X if

(Z) for any $\varepsilon > 0$ there exists a continuous mapping $f : X \to X$ such that

$$f(X) \cap B = \emptyset \quad \text{and} \quad \tilde{d}(f, \mathrm{id}_X) = \sup\bigl\{d\bigl(f(x), x\bigr) \colon x \in X\bigr\} < \varepsilon,$$

where d is a metric in X. A subset $B \subset X$ is called a *σZ-set in X* if B is a countable union of Z-sets in X. The former definition is equivalent to the following: a subset of a Hilbert cube X is a σZ-set if it is an F_σ-set in X and condition (Z) holds [11, Lemma 6.2.2].

Let $\mathcal{M}$ be a class of spaces satisfying the following conditions:

(i) all spaces homeomorphic to an element of $\mathcal{M}$ belong to $\mathcal{M}$;
(ii) the union of any two elements of $\mathcal{M}$ belongs to $\mathcal{M}$;
(iii) if $M \in \mathcal{M}$ then all closed subsets of M belong to $\mathcal{M}$.

An embedding in a Hilbert cube is called a *Z-embedding* if it maps its domain onto a Z-set.

A subset A of a Hilbert cube X is called *strongly $\mathcal{M}$-universal* in X if for every $M \in \mathcal{M}$ with $M \subset I^\infty$ and for every compact set $K \subset I^\infty$, each embedding $f : I^\infty \to X$ which is a Z-embedding on K, can be approximated arbitrarily closely by a Z-embedding $g : I^\infty \to X$ such that $g|_K = f|_K$ while moreover $g^{-1}(A) \setminus K = M \setminus K$.

Definition 2.3. [5] Let X be a Hilbert cube. A set $A \subset X$ is called an *$\mathcal{M}$-absorber* in X provided that:

(1) $A \in \mathcal{M}$;
(2) A is contained in a σZ-set in X;
(3) A is strongly $\mathcal{M}$-universal.

Theorem 2.4. [5, Theorem 2.3] *Let X be a Hilbert cube and let A and B be $\mathcal{M}$-absorbers for X. Then there is a homeomorphism $h : X \to X$ with $h(A) = B$. Moreover, h can be chosen arbitrarily close to the identity.*

Henceforth, the symbol d will denote the standard Euclidean metric in I^n or the metric $d((x_i), (y_i)) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i|$ in I^∞.

The following two lemmas are very useful in proving that some sets are Z-sets in hyperspaces 2^{I^∞} or $C(I^\infty)$.

Lemma 2.5. [5] *Let $n \in \mathbb{N} \cup \{\infty\}$. There exists a continuous deformation*

$$H_0 : 2^{I^n} \times I \to 2^{I^n}$$

such that, for any $(A,t) \in 2^{I^n} \times (0, \frac{1}{2}]$,

(1) *the set $H_0(A,t)$ is finite,*
(2) *$d_H(A, H_0(A,t)) \leqslant 2t$,*
(3) *$H_0(A,t) \subset [t, 1-t]^n$.*

The deformation through finite sets described in Lemma 2.5 can further be modified by connectifying its images. We thus get a deformation through graphs as follows.

Lemma 2.6. [5] *Let $n \in \mathbb{N} \cup \{\infty\}$, $n \geqslant 2$. There exists a continuous deformation*

$$H : 2^{I^n} \times \left[0, \frac{1}{2}\right] \to 2^{I^n}$$

such that, for any $(A,t) \in 2^{I^n} \times [0, \frac{1}{2}]$,

(1) *if $t > 0$, then $H(A,t)$ is a graph which is connected if A is connected and whose edges are segments of lengths $\leqslant 4t$,*
(2) *if $t > 0$ and x, y are vertices of graph $H(A,t)$ such that $\|x - y\| \leqslant 4t$, then the segment $\overline{xy}$ is an edge of $H(A,t)$,*
(3) *$d_H(A, H(A,t)) \leqslant 4t$,*
(4) *$H(A,t) \subset [t, 1-t]^n$,*
(5) *if $t > 0$ and $i \leqslant n$, then the numbers $a_i = \min\{x_i\colon (x_1, x_2, \ldots) \in H(A,t)\}$ and $b_i = \max\{x_i\colon (x_1, x_2, \ldots) \in H(A,t)\}$ are the ith coordinates of some vertices of $H(A,t)$.*

Proof. Using homotopy H_0 from Lemma 2.5, define $H : 2^{I^n} \times [0, \frac{1}{2}] \to 2^{I^n}$ by

$$H(A,t) = H_0(A,t) \cup \bigcup_{a,b \in H_0(A,t)} \big(\overline{ab} \cap \big(\overline{B}(a; 2t) \cup \overline{B}(b; 2t)\big)\big),$$

where $\overline{B}(a; \alpha)$ denotes the closed α-ball in I^n centered at point a (see [5] for details). □

3. Strong universality

3.1. Auxiliary embedding θ

For $q = (q_i) \in I^\infty$, let

$$r_i(q) = 4^{-(i+1)}(1 + q_i), \qquad a_i = \big(-1 + 2^{-i}, 0\big) \in \mathbb{R}^2$$

and

$$\theta_0(q) = \big([-1,0] \times \{0\}\big) \cup O\left(\left(-\frac{1}{2}, 0\right); \frac{1}{2}\right) \cup \bigcup_{i=1}^{\infty} O\big(a_i; r_i(q)\big) \subset [-1,0] \times [-1,1],$$

where $O(a; r)$ denotes the circle in $\mathbb{R}^2$ of radius r centered at a. The set $\theta_0(q)$ is a hereditarily decomposable continuum which is the union of countably many mutually disjoint circles and the diameter segment of the largest circle. Observe that the mapping

$$\theta_0 : I^\infty \to C\big([-1,0] \times [-1,1]\big)$$

is a continuous embedding. Moreover, if $i \neq j$ and $q, q' \in I^\infty$ then $O(a_i, r_i(q)) \cap O(a_j, r_j(q')) = \emptyset$ and $r_i(I^\infty) \cap r_j(I^\infty) = \emptyset$.

Put $\theta = \theta_0$ if $n = 2$. Otherwise define an embedding

$$\theta : I^\infty \to C\big([-1,0] \times [-1,1] \times I^{n-2}\big) \tag{3.1}$$

by $\theta(q) = \theta_0(q) \times \{\mathbf{0}\}$, where all the coordinates of $\mathbf{0}$ are equal to 0.

Remark 3.1. In the following the sets $[-1,0] \times [-1,1] \times I^{n-2}$ and I^n are considered as subspaces of $[-1,1]^n$. The addition and the scalar multiplication that appear in the formulas below are understood as usual linear operations. Observe that if $\varepsilon \in (0, \frac{1}{2})$, $A \subset [\varepsilon, 1-\varepsilon]^n$ and $C \subset [-1,1]^2 \times I^{n-2}$, then the set $A + \varepsilon C = \{a + \varepsilon c\colon a \in A,\ c \in C\}$ is a subset of $[0,1]^n$.

The following lemma will be used to get the strong Π^1_1-universality property.

3.2. A sufficient condition

Lemma 3.2. *Let $\mathcal{M}$ be a class of spaces satisfying conditions* (i)–(iii). *Let $n \in \mathbb{N} \cup \{\infty\}$, $n \geqslant 2$, $\mathcal{S} \in \{C(I^n), 2^{I^n}\}$ and θ be the embedding from* (3.1). *Suppose $\mathcal{A}$ is a subset of $\mathcal{S}$ such that*

(1) $\mathcal{A} \in \mathcal{M}$,
(2) *for an arbitrary set $M \subset I^\infty$, $M \in \mathcal{M}$, there exists a continuous mapping $\xi : I^\infty \to \mathcal{S}$ such that $\xi^{-1}(\mathcal{A}) = M$ and*
(3) *for every $\varepsilon \in (0, \frac{1}{2}]$, for each graph $\Gamma \in \mathcal{S}$, $\Gamma \subset [\varepsilon, 1-\varepsilon]^n$ with straight line edges, for each nonempty subset T of vertices of Γ and for each continuum $C \in \theta(I^\infty)$, the union*

$$\Gamma \cup \bigcup_{v \in T} \big(v + \varepsilon C + \varepsilon \xi(x)\big)$$

belongs to $\mathcal{A}$ if and only if $x \in M$.

Then $\mathcal{A}$ is strongly $\mathcal{M}$-universal in $\mathcal{S}$.

Proof. Let $M \subset I^\infty$, $M \in \mathcal{M}$ and let $F : I^\infty \to C(I^n)$ be an embedding which is Z-embedding on a compact set $K \subset I^\infty$. Fix $\varepsilon > 0$. Let $\mu(q) = \frac{1}{12}\min\{\varepsilon, d_H(F(q), F(K))\}$. Define $G : I^\infty \to \mathcal{S}$ as follows:

$$G(q) = H\big(F(q), \mu(q)\big) + \bigcup_{x \in H_0(F(q),\mu(q))} \big(x + \mu(q)\theta(q) + \mu(q)\xi(q)\big) \tag{3.2}$$

where $H : \mathcal{S} \times [0, \frac{1}{2}] \to \mathcal{S}$ is a homotopy described in Lemma 2.6 or its restriction to $C(I^n) \times [0, \frac{1}{2}]$. Observe that

$$G|_K = F|_K, \tag{3.3}$$

$$d_H\big(G(q), F(q)\big) < 6\mu(q) \leqslant \frac{1}{2}\min\big\{\varepsilon, d_H\big(F(q), F(K)\big)\big\},$$

$$G^{-1}(\mathcal{A}) \setminus K = M \setminus K. \tag{3.4}$$

Injectivity. Obviously $G|_K$ is injective because $G|_K = F|_K$ and F is an embedding.

If $x, y \in I^\infty \setminus K$ and $G(x) = G(y)$, then $\mu(x)$ and $\mu(y)$ are positive. Pick a point $p = (p_1, \ldots, p_n)$ of a continuum $G(x) = G(y)$ with $p_1 = \min\{z_1\colon (z_1, \ldots, z_n) \in G(x)\}$. The point p belongs to exactly one circle contained in $G(x) = G(y)$. This circle is of diameter $\mu(x) = \mu(y)$ and it is a subset of continua $v + \mu(x)\theta(x)$ and $v + \mu(y)\theta(y)$, where $v = p + \mu(x)(1, 0, \ldots, 0)$ is a vertex of the graphs $H(F(x), \mu(x))$ and $H(F(y), \mu(y))$. By the definition of G and by the choice of p, it follows that the circle whose diameter segment is $\overline{pv}$ is disjoint from every copy of $\xi(x)$ in $G(x)$, i.e. every set of the form $w + \mu(x)\xi(x)$ where $w \in H_0(F(x), \mu(x))$. Let $\mathcal{O}$ denote the family of all circles contained in $G(x)$ with centers in $\overline{pv}$. The graph $H(F(x), \mu(x))$ contains no circle, so every element of $\mathcal{O}$ must be contained in a copy of $\theta(x)$, i.e. in a set of the form $w + \mu(x)\theta(x)$ for some $w \in H_0(F(x), \mu(x))$. Consider a subfamily $\mathcal{O}_i \subset \mathcal{O}$ of all the circles about points $p + 2^{-i}\mu(x)$ and of radii in $[4^{-(i+1)}\mu(x), 4^{-(i+\frac{1}{2})}\mu(x)]$. Observe that $\mathcal{O}_i \neq \emptyset$; e.g. the circle of radius $\mu(x)r_i$ is an element of $\mathcal{O}_i$. Since $r_i(I^\infty) \cap r_j(I^\infty) = \emptyset$ for $i \neq j$, $\mathcal{O}_i$ is degenerate. Therefore the structure of $\mathcal{O}$ carries a full information about $r_i(x)$ for every $i \in \mathbb{N}$. Applying the same argument to $G(y)$, one can identify $r_i(y)$ for each $i \in \mathbb{N}$. Thus, for $G(x) = G(y)$ and an arbitrary $i \in \mathbb{N}$, the numbers $r_i(x)$ and $r_i(y)$ are equal. It follows that $\theta(x) = \theta(y)$ and, by injectivity of θ, $x = y$.

Suppose now that $x \in K$, $y \in I^\infty \setminus K$ and $G(x) = G(y)$. Then, by (3.3), $G(y) = F(x) \in F(K)$. By the triangle inequality, we have

$$d_H\big(F(y), F(K)\big) \leqslant d_H\big(F(y), G(y)\big) + d_H\big(G(y), F(K)\big),$$

so

$$d_H\big(G(y), F(K)\big) \geqslant d_H\big(F(y), F(K)\big) - d_H\big(F(y), G(y)\big).$$

Then, by (3.4) and the fact that F is one-to-one,

$$d_H\big(G(y), F(K)\big) > \frac{1}{2} d_H\big(F(y), F(K)\big) > 0,$$

which contradicts $G(y) \in F(K)$.

$G(I^\infty)$ is a Z-set. By the assumption, $G(K) = F(K)$ is a Z-set. The set $G(I^\infty) \setminus G(K)$ is open in the compact space $G(I^\infty)$, so $I^\infty \setminus K$ is a countable union of compact sets A_n. Observe that every element of $G(I^\infty) \setminus G(K)$ contains free arcs. Given a number $\delta > 0$, define the function $b : C(I^n) \to C(I^n)$ putting $b(A) = \{x \in I^n \colon d(x, A) \leqslant \delta\}$. Observe that $b(G(I^\infty \setminus K)) \cap G(I^\infty \setminus K) = \emptyset$ and $\tilde{d}(b, \mathrm{id}_{C(I^n)}) \leqslant \delta$. Therefore $G(I^\infty) \setminus G(K)$ and $G(K)$ are σZ-sets, so $G(I^\infty)$ is a σZ-set. Since $G(I^\infty)$ is compact, it is a Z-set in $C(I^n)$. $\square$

Remark 3.3. The imbedding θ in 3.2 can be replaced by any other continuous function that provides injectivity of $G|_{I^\infty \setminus K}$ for G defined as in (3.2).

3.3. Application to $\mathcal{HD}(I^n)$ and $\mathcal{NP}(I^n)$

Lemma 3.4. *For every coanalytic set $M \subset I^\infty$, there exists a continuous mapping $\xi : I^\infty \to \mathcal{D}(I^n)$ such that $\xi(q)$ is a countable union of arcs if $q \in M$ and $\xi(q)$ contains a pseudoarc if $q \notin M$.*

Proof. Fix a coanalytic set $M \subset I^\infty$. Let $S : \mathbb{N}^{<\mathbb{N}} \to 2^{I^\infty}$ be as in Lemma 2.2 for $A = I^\infty \setminus M$. For $\tau \in \mathbb{N}^{<\mathbb{N}}$, let $\lambda_\tau : I^\infty \to [0, 1]$ be a continuous function defined by

$$\begin{aligned} &\lambda_\tau(q) = 1 && \text{for } q \in S(\tau), \\ &\lambda_\tau(q) = 0 && \text{for } q \notin \operatorname{int} S\big(\tau \restriction (|\tau| - 1)\big), \\ &0 < \lambda_\tau(q) < 1 && \text{for } q \in \operatorname{int} S\big(\tau \restriction (|\tau| - 1)\big) \setminus S(\tau). \end{aligned} \tag{3.5}$$

Define a family $\mathcal{I} = \{I_\tau \subset I \colon \tau \in \mathbb{N}^{<\mathbb{N}}\}$ of segments and continuous mappings $J_\tau : I^\infty \to C(I)$ and $L_\tau : I^\infty \to C(I^2)$ as follows. Put

$$J_{\langle i\rangle}(q) = I_{\langle i\rangle} = \big[2^{-(2i-1)}, 2^{-2(i-1)}\big] \quad \text{and} \quad L_{\langle i\rangle}(q) = I \times I_{\langle i\rangle}.$$

Suppose segments I_τ and functions J_τ, L_τ have been defined for all sequences τ of length $\leqslant n$ such that

- $J_\tau(q)$ is a segment or a point;
- $L_\tau(q)$ is an arc or a uniform band whose left side is $\{0\} \times J_\tau(q)$;
- if $\tau < \tau'$ then $I_{\tau'} \subset I_\tau$ and $L_{\tau'}(q) \subset L_\tau(q)$;
- if $|\tau| = |\tau'|$ and $\tau \neq \tau'$, then $I_{\tau'} \cap I_\tau = \emptyset$ and $L_{\tau'}(q) \cap L_\tau(q) = \emptyset$;
- if $L_\tau(q)$ with $|\tau| \geqslant 2$ is a band then it is crooked in $L_{\tau \restriction (|\tau|-1)}(q)$ according to the pattern for $B_{|\tau|-1}$ in the Knaster's construction.

Let η be a sequence of length $n + 1$. We have $\eta = \langle \eta \restriction n, j\rangle$, for some $j \in \mathbb{N}$. If $J_{\eta\restriction n}(q)$ is a point then put $J_\eta(q) = J_{\eta\restriction n}(q)$ and $L_\eta(q) = L_{\eta\restriction n}(q)$. Otherwise, inscribe in $L_{\eta\restriction n}(q)$ the band $P_{\eta\restriction n}(q)$ following the pattern for $B_{|\eta|-1}$ in the Knaster's construction. Denote the left side of the new band by $\{0\} \times K_{\eta\restriction n}(q)$. Let $h_{\eta\restriction n} : I \to I_{\eta\restriction n}$, $g_{\eta\restriction n,q} : I \to K_{\eta\restriction n}(q)$ and $u_{\eta\restriction n,q} : I \to [0, \lambda_{\eta\restriction n}(q)]$ be non-decreasing linear surjections. Put $I_\eta = h_{\eta\restriction n}(I_j)$ and

$J_\eta(q) = g_{\eta\restriction n}(u_{\eta\restriction n,q}(I_j))$. Define $L_\eta(q)$ as the uniform band whose left side is $\{0\} \times J_\eta(q)$ and whose borders are parallel to the borders of $P_{\eta\restriction n}(q)$.

It could seem that the band $P_{\eta\restriction n}(q)$ does not depend directly on q. However, the width of $L_{\eta\restriction 1}(q)$ varies continuously with q and $P_{\eta\restriction 1}(q)$ is constructed by division $L_{\eta\restriction 1}(q)$ into 25 rectangles and composing the new band of them, according to the Knaster's algorithm. Then, the band $L_{\eta\restriction 2}(q)$ is inscribed into $P_{\eta\restriction 1}(q)$. This step of the construction is also determined by q, since $L_{\eta\restriction 1}(q)$, $P_{\eta\restriction 1}(q)$ and the function $\lambda_{\eta\restriction 1}(q)$ depend continuously on q. The similar argument shows the continuity of $L_{\eta\restriction n}(q)$ and $P_{\eta\restriction n}(q)$ for every $n \in \mathbb{N}$.

For every $\tau \in \mathbb{N}^{<\mathbb{N}}$, denote the lower border of $P_\tau(q)$ by $D_\tau(q)$. Observe that $d_H(D_\tau(q), L_{\langle\tau,k\rangle}(q))$ tends to 0 if $k \to \infty$.

For each sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$ and natural number n, consider mappings $\xi_\sigma^n : I^\infty \to C(I^2)$ and $\phi_\sigma^n : I^\infty \to C(I^2)$ defined by

$$\phi_\sigma^n(q) = \big(I \times \{0\}\big) \cup \big(\{0\} \times I\big) \cup \bigcup_{i=1}^{n} D_{\sigma\restriction i}(q),$$

$$\xi_\sigma^n(q) = \phi_\sigma^n(q) \cup L_{\sigma\restriction n}(q),$$

Clearly, $\xi_\sigma^{n+1}(q) \subset \xi_\sigma^n(q)$. Put

$$\phi^n(q) = \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \phi_\sigma^n(q),$$

$$\xi^n(q) = \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \xi_\sigma^n(q),$$

$$\xi(q) = \bigcap_{n=1}^{\infty} \xi^n(q)$$

(see Fig. 4).

Observe that mappings $\xi^n : I^\infty \to C(I^2)$ and $\phi^n : I^\infty \to C(I^2)$ are continuous and

$$\phi^n(q) \subset \xi(q) \subset \xi^n(q),$$

$$d_H\big(\xi^n(q), \xi^{n+1}(q)\big) \leqslant 2^{-n},$$

$$d_H\big(\xi^n(q), \phi^n(q)\big) \leqslant 2^{-n}.$$

It follows that the sequence $(\xi^n)_{n\in\mathbb{N}}$ uniformly converges to the continuous mapping ξ. Recall that, for distinct $\sigma_1, \sigma_2 \in \mathbb{N}^{\mathbb{N}}$ and k big enough, bands $L_{\sigma_1\restriction k}(q)$ and $L_{\sigma_2\restriction k}(q)$ are disjoint. Therefore

$$\xi(q) = \bigcap_{n=1}^{\infty} \xi^n(q) = \big(I \times \{0\}\big) \cup \big(\{0\} \times I\big) \cup \bigcup_{n=1}^{\infty} \phi^n(q) \cup \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \big(L_{\sigma\restriction n}(q)\big). \tag{3.6}$$

Obviously, $\xi(q)$ is a decomposable continuum for every $q \in I^\infty$. We are going to show that $\xi(q)$ contains a pseudoarc if and only if $q \in M$.

Suppose first that $q \notin M$. There exists a sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $q \in S(\sigma \restriction n)$ for each $n \in \mathbb{N}$. Thus, $\lambda_{\sigma\restriction n}(q) = 1$ for each n. Then $(L_{\sigma\restriction n}(q))_{n\in\mathbb{N}}$ is a descending sequence of bands each of them being inscribed in the previous one according to the appropriate pattern in Knaster's construction. Thus the intersection of the bands is a pseudoarc contained in $\xi(q)$.

If $q \in M$, then, for each $\sigma \in \mathbb{N}^{\mathbb{N}}$, there exists n such that $q \notin S(\sigma \restriction n)$, i.e., the sequence $(\lambda_{\sigma\restriction n}(q))_{n\in\mathbb{N}}$ eventually equals 0. In this case, for n big enough, $D_{\sigma\restriction n}(q) = L_{\sigma\restriction n}(q) = L_{\sigma\restriction(n+1)}(q) = L_{\sigma\restriction(n+2)}(q) = \cdots$, in particular $L_{\sigma\restriction n}(q)$ is an arc. Therefore, one can replace infinite sequences σ in formula (3.6) with finite sequences which shows that the right-hand side of (3.6) is the countable union of arcs. □

Corollary 3.5. *If $n \in \mathbb{N} \cup \{\infty\}$ and $n \geqslant 2$ then $\mathcal{HD}(I^n)$ and $\mathcal{NP}(I^n) \cap \mathcal{D}(I^n)$ are strongly Π_1^1-universal in $C(I^n)$.*

Proof. Observe that if $\mathcal{A} = \mathcal{HD}(I^n)$ or $\mathcal{A} = \mathcal{NP}(I^n) \cap \mathcal{D}(I^n)$ then $\mathcal{A}$ and the mapping ξ from Lemma 3.4 satisfy hypotheses of Lemma 3.2 for $\mathcal{M} = \Pi_1^1$. □

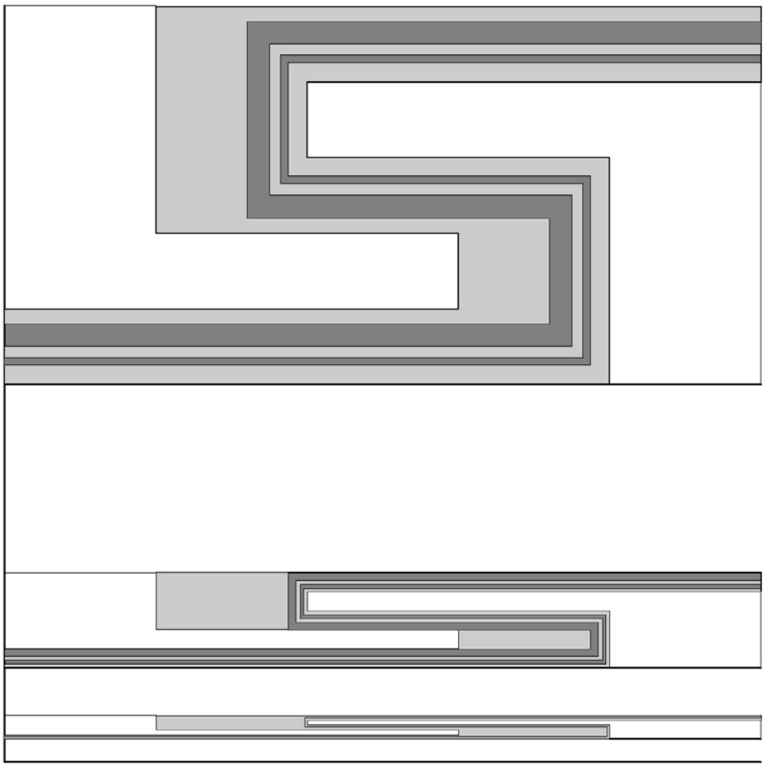

Fig. 4. $\xi^2(x)$.

4. Covering by σZ-sets

Clearly, both the hyperspaces $\mathcal{HD}(I^n)$ and $\mathcal{NP}(I^n) \cap \mathcal{D}(I^n)$ are contained in $\mathcal{D}(I^n)$. It is known—and easy to check—that $\mathcal{D}(X) = \bigcup_{j \in \mathbb{N}} \mathcal{B}_j$ where

$$\mathcal{B}_j = \{K \in C(X)\colon (\exists_{L,M \in C(X)})(K = L \cup M \wedge L \not\subseteq N(M, 2^{-j}) \wedge M \not\subseteq N(L, 2^{-j}))\}.$$

Note that all the sets $\mathcal{B}_j$ are compact. We prove that they are Z-sets if $X = I^n$, $3 \leqslant n \leqslant \infty$.

Fix $j \in \mathbb{N}$ and $\varepsilon > 0$. Put $\delta = \frac{1}{4}\min\{\varepsilon, 2^{-j}\}$. We will find a continuous function $T : C(I^n) \to C(I^n) \setminus \mathcal{B}_j$ such that $\tilde{d}_H(T, \mathrm{id}_{C(I^n)}) < \varepsilon$.

The following construction is described in [5, §4], where it is employed to show that the space of locally connected subcontinua of I^n ($n \geqslant 3$) is contained in a σZ-set in $C(I^n)$. The same mapping T can be applied to prove that $\mathcal{B}_j$ is a Z-set for every $j \in \mathbb{N}$.

Lemma 4.1. [3,5] *For every $\delta > 0$ there is a finite connected graph $\Gamma \subset (0,1)^n$ and a continuous function $\psi : C(I^n) \to C(\Gamma)$ with $\tilde{d}_H(\psi, \mathrm{id}_{C(I^n)}) < \delta$.*

Lemma 4.2. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $n \geqslant 3$. For every finite connected graph $\Gamma \subset (0,1)^n$ and $\delta > 0$ there is a continuous function $T_0 : C(\Gamma) \to C(I^n)$ such that, for every subgraph $A \in C(\Gamma)$,*

(1) *$T_0(A)$ is a one-dimensional continuum*;
(2) $A \subset T_0(A) \subset N(A, 2\delta)$;
(3) *if v is a vertex of A that is not an endpoint of A then $T_0(A) \setminus \{v\}$ is the union of disjoint sets X_v and Y_v where*
- $A \subset X_v \cup \{v\}$;
- $A \subset \mathrm{cl}\, Y_v$;

- *$X_v \cup \{v\}$ and Y_v are connected;*
- *if D is a continuum contained in $X_v \cup Y_v$ then either $D \subset X_v$ or $D \subset Y_v$.*

Proof. Let $T_0 = T_m$ for T_m constructed in [5, §4] and m big enough. □

Lemma 4.3. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $n \geqslant 3$. Then, for every $j \in \mathbb{N}$, $\mathcal{B}_j$ is a Z-set.*

Proof. Define $T : C(I^n) \to C(I^n)$ by $T = T_0 \circ \psi$ with ψ and T_0 defined in Lemmas 4.1 and 4.2. Then $\tilde{d}_H(T, \mathrm{id}_{C(I^n)}) < 3\delta < \varepsilon$.

Now check that $T(C(I^n)) \cap \mathcal{B}_j = \emptyset$. Pick an arbitrary subcontinuum C of I^n. Denote $\psi(C)$ by A. Suppose that $T(C) \in \mathcal{B}_j$. Then there are continua $K, L \subset I^n$ such that $T(C) = K \cup L$, $K \setminus N(L, 2^{-j}) \neq \emptyset$ and $L \setminus N(K, 2^{-j}) \neq \emptyset$. Recall that $T(C) = T_0(A) \subset N(A, 2\delta)$ (Lemma 4.2). Thus K and L must touch A; moreover, $(K \cap A) \setminus N(L, 2\delta) \neq \emptyset$ and $(L \cap A) \setminus N(K, 2\delta) \neq \emptyset$ (otherwise $K \subset N(L, 4\delta) \subset N(L, 2^{-j})$ or $L \subset N(K, 4\delta) \subset N(K, 2^{-j})$). We can assume that the vertices of Γ form an ε-net in Γ. Since $\delta \leqslant 2^{-(j+2)}$, we can choose a vertex v of the graph Γ so that v is not an endpoint of graph A, $v \in K \setminus L$ and $d(v, L) \geqslant \delta$. Decompose $T(C) \setminus \{v\}$ into disjoint sets X_v and Y_v according to Lemma 4.2. Since $v \notin L$, we have $L \subset X_v$ or $L \subset Y_v$. In the former case $A \subset \mathrm{cl}\, Y_v \subset K$, so $L \subset T(C) = T_0(A) \subset N(A, 2\delta) \subset N(K, 2\delta)$. In the latter case $A \subset X_v \cup \{v\} \subset K$ and, similarly, $L \subset N(K, 2\delta)$.

Therefore $T(C(I^n))$ and $\mathcal{B}_j$ are disjoint. □

Corollary 4.4. *If $n \in \mathbb{N} \cup \{\infty\}$ and $n \geqslant 3$ then $\mathcal{D}(I^n)$ is a σZ-set in $C(I^n)$.*

By [12, Corollary 5.3.10] and [2, Lemmas 2.3 and 2.5] we obtain the following interesting corollary.

Corollary 4.5. *If $n \in \mathbb{N} \cup \{\infty\}$ and $n \geqslant 3$ then there is a deformation of $C(I^n)$ through $C(I^n) \setminus \mathcal{D}(I^n)$. In other words, $C(I^n) \setminus \mathcal{D}(I^n)$ is homotopy dense in $C(I^n)$. In particular, $C(I^n) \setminus \mathcal{D}(I^n)$ is an absolute retract and for every $\varepsilon > 0$ there is a continuous function that is ε-close to $\mathrm{id}_{C(I^n)}$ and sends every continuum onto an indecomposable continuum.*

Question 4.6. Is $\mathcal{D}(I^2)$ a σZ-set in $C(I^2)$?

Question 4.7. Suppose that $n \in \mathbb{N} \cup \{\infty\}$, $n \geqslant 2$. Is $\mathcal{D}(I^n)$ homeomorphic to the pseudoboundary of the Hilbert cube?

Question 4.8. Suppose that $n \in \mathbb{N} \cup \{\infty\}$, $n \geqslant 2$. Is $C(I^n) \setminus \mathcal{D}(I^n)$ homeomorphic to l_2?

5. Main results

Theorem 5.1. *Let $n \in \mathbb{N} \cup \{\infty\}$ and $n \geqslant 3$. Then the hyperspaces $\mathcal{HD}(I^n)$ and $\mathcal{NP}(I^n) \cap \mathcal{D}(I^n)$ are Π^1_1-absorbers in the Hilbert cube $C(I^n)$.*

Proof. By Lemma 2.1 both the hyperspaces are coanalytic. Thus, by Corollaries 3.5 and 4.4, they are Π^1_1-absorbers in $C(I^n)$. □

Remark 5.2. R. Cauty [2] proved that $\mathcal{H}$ is a Π^1_1-absorber. However, the careful reader can notice that the above definition of $\mathcal{M}$-absorber (Definition 2.3) differs from the one in [2]. Recall that, by Lemma 2.5, there is a deformation of 2^I through $\mathcal{H}$. Therefore, it follows from [2, Lemme 2.6] and [1, Theorem 3.1.3] that $\mathcal{H}$ is a Π^1_1-absorber in 2^I.

Lemma 5.3. *The Hurewicz set $\mathcal{H}$ is a Π^1_1-absorber in 2^I.*

From Theorem 5.1, Theorem 2.4 and Lemma 5.3, we get the following corollary.

Corollary 5.4. *Let $n \in \mathbb{N} \cup \{\infty\}$, $n \geqslant 3$. Then*

(1) *Each of the hyperspaces* $\mathcal{HD}(I^n)$ *and* $\mathcal{NP}(I^n) \cap \mathcal{D}(I^n)$ *is homeomorphic to the Hurewicz set* $\mathcal{H}$.
(2) *The complements of the hyperspaces, i.e. the sets* $C(I^n) \setminus \mathcal{HD}(I^n)$ *and* $C(I^n) \setminus (\mathcal{NP}(I^n) \cap \mathcal{D}(I^n))$ *are homeomorphic to* $2^I \setminus \mathcal{H}$.

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